

Time dependent quantum generators for the Galilei group

G. Filippelli^{1, a)}*Dipartimento di Matematica, Università della Calabria*

In 1995 Doebner and Mann introduced an approach to the ray representations of the Galilei group in $(1+1)$ -dimensions, giving rise to quantum generators with an explicit dependence on time. Recently (2004) Wawrzycki proposed a generalization of Bargmann's theory: in his paper he introduce phase exponents that are explicitly dependent by 4-space point. In order to find applications of such generalization, we extend the approach of Doebner and Mann to higher dimensions: as a result, we determine the generators of the ray representation in $(2+1)$ and $(3+1)$ dimensions. The differences of the outcoming formal apparatus with respect to the smaller dimension case are established.

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I. INTRODUCTION

In 2004 Wawrzycki proposes a generalization of Bargmann's theory¹, introducing the phase exponents that explicitly depends on time, for non-relativistic groups, and phase exponents that explicitly depends on 4-space point, for relativistic groups.

On the other hand, in 1995, Doebner and Mann², studying Galilei group in $(1+1)$ dimensions, calculated the generators group's representations that explicitly depends on time, while for Galilei group in $(2+1)$ dimensions they founded the generators group's representations just calculated by Bose³ and Grigore⁴.

At this point, in order to generalize Doebner and Mann work about higher dimensions, in Section III we calculate the generators of the Galilei group's representations and we show that they explicitly depends on time. These representations are valid for $(2+1)$ and $(3+1)$ dimensions. At the end, from these representations, we calculate the ray representations of Galilei group, and we show that they explicitly depends on time.

First of all, we introduce the ray representations of a given Lie group (Section IA), and we briefly remind the application of the Bargmann's theory⁵ to Galilei group (Section II). In the Section III we find the time dependence of the ray representations of galilean group in $(3+1)$ and $(2+1)$ dimensions, with phase exponents that depend on time, and we propose a physical interpretation either for it, either for Bargmann's phase exponent.

A. Ray representations

The vector ray notion, introduced by Weyl⁶, can be extended also to the operators, in particular to unitary operators. According to Wigner's theorem⁷⁻⁹, every symmetry's transformation T_r can be represented by an unitary (or anti-unitary) operator, which is unique up to a

phase factor, id est T_r can be represented by an unitary operator ray defined by:

$$\mathcal{U}_r = \{e^{i\theta} U_r, \theta \in \mathbb{R}\}. \quad (1)$$

So, \mathcal{U}_e is the ray containing the identity operator and \mathcal{U}_r^{-1} is the inverse of \mathcal{U}_r , i.e. the ray containing the operators U_r^{-1} inverse of $U_r \in \mathcal{U}_r$,

$$\mathcal{U}_r^{-1} = \{V_r, \text{ such that } V_r = U_r^{-1}, U_r \in \mathcal{U}_r\}$$

then $\mathcal{U}_r \cdot \mathcal{U}_r^{-1} = \mathcal{U}_r^{-1} \cdot \mathcal{U}_r = \mathcal{U}_e$.

Now, let G be a symmetry group with elements r, s, t, \dots ; then, for a given choice of the unitary operator $U_r \in \mathcal{U}_r$ representing the elements of G , in general the following composition law holds:

$$U_r U_s = \omega(r, s) U_{rs}$$

that we can rewrite for the operator rays as:

$$\mathcal{U}_r \cdot \mathcal{U}_s = \mathcal{U}_{rs}$$

where $\omega(r, s)$ is a factor of modulus 1 ($|\omega(r, s)| = 1$), $r, s \in \mathcal{N}_0 \subset G$, and \mathcal{N}_0 is a neighbourhood of G identity.

The correspondence $r \rightarrow \mathcal{U}_r$ realizes a **ray representation** (or *projective representation*) of G . It is equivalent to an usual unitary representation of G if a correspondence $r \rightarrow U_r \in \mathcal{U}_r$ exists such that $\omega(r, s) \equiv 1$, $\forall r, s \in G$.

In order to classify the ray representations of a given Lie group G , it is sufficient classify the phase factor equivalence classes; indeed, for a different admissible set of representatives¹⁰ $U'_r = \phi(r) U_r$, we obtain

$$U'_r \cdot U'_s = \omega'(r, s) U'_{rs}$$

where

$$\omega'(r, s) = \omega(r, s) \frac{\phi(r)\phi(s)}{\phi(rs)}. \quad (2)$$

But it is more advantageous to replace (local) factors with (local) exponents by setting $\omega(r, s) = e^{i\xi(r, s)}$. So, a phase exponent of a group G is a real valued continuous

^{a)} Also at Istituto Nazionale di Fisica Nucleare, Gruppo collegato di Cosenza

Electronic mail: gianluigi.filippelli@gmail.com

function $\xi(r, s)$ which is defined for all r, s in G and which satisfies the relations⁵:

$$\xi(e, e) = 0 \quad (3a)$$

$$\xi(r, s) + \xi(rs, t) = \xi(s, t) + \xi(r, st) \quad (3b)$$

For every phase exponent defined on the group (or on a neighbourhood), a function called *infinitesimal exponent* Ξ defined on the Lie algebra of the group exists, that is in one-to-one linear correspondence with phase exponent:

$$\begin{aligned} \Xi(a, b) = \lim_{\tau \rightarrow 0} \tau^{-2} & \left(\xi((\tau a)(\tau b), (\tau a)^{-1}(\tau b)^{-1}) + \right. \\ & \left. + \xi(\tau a, \tau b) + \xi((\tau a)^{-1}, (\tau b)^{-1}) \right). \end{aligned} \quad (4)$$

II. THE GALILEI'S GROUP

Now, we remind the study about the Galilei's group, its Lie algebra and ray representations in $(3+1)$ -dimensions (Section II A), $(2+1)$ -dimensions (Section II B) and $(1+1)$ -dimensions (Section II C). In the last case we show how Doebner and Mann² calculate the generators of Galilean group, which turn out to depend on time, and we propose the Galilei's group's ray representations that depend on time.

A. The Galilei's group in $(3+1)$ -dimensions

The Galilean group is constituted by all space-time transformations from an inertial reference frame to another one. The most general Galilean transformation of the Galilean group G is:

$$\begin{cases} x' = Wx + vt + u \\ t' = t + \eta \end{cases} \quad (5a)$$

where x', x are spatial vectors, v is the relative velocity, u is a space translation, t is time and η a time translation, with W an orthogonal transformation (e.g. rotation).

In order to classify the ray representations of Galilei's group, we can represent, following Bargmann⁵, the generic element r of the Galilean group G as:

$$r = (W_r, \eta_r, v_r, u_r). \quad (5b)$$

So, the group multiplication is given by

$$\begin{aligned} rs &= (W_r, \eta_r, v_r, u_r) \cdot (W_s, \eta_s, v_s, u_s) = \\ &= (W_r W_s, \eta_r + \eta_s, W_r v_s + v_r, W_r u_s + u_r + \eta_s v_r) \end{aligned} \quad (6)$$

Now, to classify the ray representations of G , we first must describe the algebra of Galilei's group: algebra standard basis is constituted by a_{ij} , anti-symmetric 3×3 matrix where only elements ij are non-null; b_i , the translations generator; d_i , the pure galilean transformations generator ($1 \leq i \leq n$, $1 \leq j \leq n$, where n is the space

dimension); f , the time translations generator. The generators algebra is given by:

$$[a_{ij}, a_{kl}] = \delta_{jk} a_{il} - \delta_{ik} a_{jl} + \delta_{il} a_{jk} - \delta_{jl} a_{ik} \quad (7a)$$

$$[a_{ij}, b_k] = \delta_{jk} b_i - \delta_{ik} b_j; \quad [b_i, b_j] = 0 \quad (7b)$$

$$[a_{ij}, d_k] = \delta_{jk} d_i - \delta_{ik} d_j; \quad [d_i, d_j] = 0; \quad [d_i, b_j] = 0 \quad (7c)$$

$$[a_{ij}, f] = 0; \quad [b_k, f] = 0; \quad [d_k, f] = b_k \quad (7d)$$

At this point we can calculate all infinitesimal exponent of the Galilei's group. The only non-null exponent is:

$$\Xi(b_i, d_k) = -\Xi(d_k, b_i) = \gamma \delta_{ik}. \quad (8)$$

The corresponding phase exponent ξ is a multiple of the function ξ_0 :

$$\begin{aligned} \xi(r, s) = \gamma \xi_0(r, s) = \frac{1}{2} \gamma & (\langle u_r | W_r v_s \rangle + \\ & - \langle v_r | W_r u_s \rangle + \eta_s \langle v_r | W_r v_s \rangle), \end{aligned} \quad (9)$$

by a multiplicative factor γ , which is interpreted as the mass of a free particle¹¹.

At this point Bargmann determined the Galilean ray representations⁵:

$$\begin{aligned} \phi'(p) &= U_r \phi(r^{-1}p) = \\ &= e^{-i(\langle p | u \rangle - \frac{\eta}{2\gamma} \langle p | p \rangle + \frac{\eta}{2} \gamma \langle v | v \rangle - \gamma \langle u | v \rangle)} \phi(r^{-1}p) \end{aligned} \quad (10)$$

where the functions $\phi(p)$ are the wave functions in the *Heisenberg representation*¹² ($\phi(p) \in \mathcal{L}_2(\mathbb{R}^2, p)$), and p being the linear momentum.

B. The Galilei's group in $(2+1)$ -dimensions

Now we can study the Galilean group in lower dimensions. By the results obtained by Bargmann⁵ about the pseudo-orthogonal groups, we must expect the emergence of new phase exponents in $(2+1)$ dimensions. Indeed, in this case, there are two non-equivalent non-trivial phase exponents^{3,4} besides ξ_0 in (9):

$$\xi_1(r, s) = \frac{1}{2} (v_r \wedge W_r v_s) \quad (11a)$$

$$\xi_2(r, s) = \theta_r \eta_s - \theta_s \eta_r \quad (11b)$$

where $(u \wedge v) = u_1 v_2 - v_1 u_2$, with u, v two-dimensional vectors.

All inequivalence classes are multiple of (11) with multiplicative factors λ and S respectively. In correspondence with different values of the three multiplicative factors γ, λ, S , we can have non-equivalent ray representations of the Galilean group. In particular, we are interested in the ray representations with $S = 0$; so we have the two following cases:

$$\begin{aligned} (U(W, \eta, v, u)f)(p) &= e^{i(\langle u | p \rangle + \frac{\eta}{2} \langle u | v \rangle + \frac{\eta}{2\gamma} \langle p | p \rangle - \frac{\lambda}{2\gamma} (v \wedge p) + s\theta)} \\ & f(W^{-1}(p + \gamma v)) \end{aligned} \quad (12)$$

$$\begin{aligned} (U(W, \eta, v, u)f)(p) &= e^{i(\langle u | p \rangle + \frac{\eta}{2} \langle u | v \rangle + \frac{\eta}{2\gamma} \langle p | p \rangle)} \\ & s(h)f(W^{-1}(p + \gamma v)) \end{aligned} \quad (13)$$

where (12) corresponds to a localizable Schrödinger system for which the *boost* representation is not abelian, and (13) describes a Schrödinger non-relativistic particle, where $s(h)$ denotes an irreducible representation of the sub-group $\{(W, 0, v, 0)\}$, and the functions $f(p) \in \mathcal{L}_2(X_r, p)$, where $X_r = \{(p_0, p)\}$, with $\langle p | p \rangle = r^2$, is an orbit.

C. The Galilei's group in $(1+1)$ -dimensions

According to Doebner and Mann², we write the generic element of the Galilean group in $(1+1)$ dimensions as

$$r = (u_r, v_r, \eta_r)$$

where u_r is a space translation, v_r a velocity translation, η_r a time translation. The corresponding phase exponent is:

$$\xi_\eta(r, s) = \frac{a_1}{2}(a_r v_s - a_s v_r + \eta_r v_r v_s) + \frac{a_2}{2}(u_r \eta_s - u_s \eta_r - \eta_r \eta_s v_r) \quad (14)$$

with $a_{1,2}$ are real numbers.

To introduce a time dependance in Hilbert space we use a kind of Hisenberg picture. For any self-adjoint operators $R(X)$ we set¹³:

$$\frac{d}{dt} R_t(X) = K R_t([H, X]) \quad (15)$$

with initial condition

$$R_{t=0}(X) = R(X)$$

(where K is a complex constant, H time translations generator, $R_t(X)$ is the time depending representation of the generator X and $\frac{\partial X}{\partial t} = 0$) we can calculate generators of Galilean group:

$$\begin{aligned} R_t(H) &= -\frac{\hbar^2}{2m} \partial_x^2 + f x + V_0, & R_t(P) &= i\hbar \partial_x - f t, \\ R_t(N) &= m x - i\hbar t \partial_x - \frac{1}{2} f t^2 \end{aligned} \quad (16)$$

where H is the time translations generator, P the space translations generator, N the boost translation generator, \hbar the Planck constant, $m = a_1$ the mass particle, $f = a_2$ an external force, $V_0 = \frac{a_3}{2a_1}$ ¹⁴.

III. EXTENSION OF DOEBNER AND MANN CALCULATION

Now we extend the Doebner and Mann approach to determine the time depending Galilean generators in $(2+1)$ and $(3+1)$ dimensions; by using these generators, we derive the unitary time depending representations for the Galilean group. As a result, besides (9), we find ray representations with a phase exponent which explicitly depends on time.

Now, applying (15) to generators of Galilei's group in $(2+1)$ - and $(3+1)$ -dimensions¹⁵, we find that $R_t(H) = R(H)$, $R_t(P_i) = R(P_i)$, $R_t(M) = R(M)$, while for N_i generators we find:

$$\begin{aligned} R_t(N_1) &= -i p_1 t + \gamma \frac{\partial}{\partial p_1} - i \frac{\lambda}{2\gamma} p_2, \\ R_t(N_2) &= -i p_2 t + \gamma \frac{\partial}{\partial p_2} - i \frac{\lambda}{2\gamma} p_1 \end{aligned}$$

for representation (12), and we find:

$$R_t(N_i) = -i p_i t + \gamma \frac{\partial}{\partial p_i}$$

for representation (13).

Now, for $v = (1, 0, v, 0) \in G$, the corresponding unitary operator time depending will be

$$\begin{aligned} \left(1 - i \langle p | v \rangle t - i \frac{\lambda}{2\gamma} (v \wedge p) + \gamma v_1 \frac{\partial}{\partial p_1} + \gamma v_2 \frac{\partial}{\partial p_2}\right) f(p) &\simeq \\ \left(1 - i \langle p | v \rangle t - i \frac{\lambda}{2\gamma} (v \wedge p)\right) &\cdot \\ \left(1 + \gamma v_1 \frac{\partial}{\partial p_1} + \gamma \frac{\partial}{\partial p_2}\right) f(p) &\simeq \\ e^{-i \langle p | v \rangle t - i \frac{\lambda}{2\gamma} (v \wedge p)} f(p + \gamma v) &= (U_t(v)f)(p) \end{aligned}$$

for representation (12), and

$$(U_t(v)f)(p) = e^{-i \langle p | v \rangle t} f(p + \gamma v)$$

for representation (13). So the representation of Galilean group, time depending, in $(2+1)$ (for $\gamma \neq 0, \lambda \neq 0, S = 0$, and for $\gamma \neq 0, \lambda = S = 0$) and $(3+1)$ dimensions is

$$(U_t(r)f)(p) = e^{-i \langle p | v_r \rangle t} (U(r)f)(p). \quad (17)$$

So, $\forall r \in G$, the corresponding $U_t(r)$ time depending unitary operator is given by the unitary operator $U(r)$ of the ray representation up to a time depending phase factor, $e^{-i \langle p | v_r \rangle t}$.

Now we can see whether the introduction of the phase $e^{i \langle p | v \rangle t}$ in the ray representation of Galilean group generates a further projective phase factor. To this purpose we calculate:

$$\begin{aligned} ((U_t(r)U_t(s))f)(p) &= e^{-i \langle p | v_r \rangle t} U_t(s)((U(r)f)(p)) \\ &= e^{-i \langle p | v_r \rangle t} e^{-i \langle W_r^{-1}(p + \gamma v_r) | v_s \rangle t} \\ &\quad \omega(r, s)(U(rs)f)(p). \end{aligned}$$

On the other hand

$$\begin{aligned} (U_t(rs)f)(p) &= e^{-i \langle p | v_{rs} \rangle t} (U(rs)f)(p) = \\ &= e^{-i \langle p | v_r \rangle t} e^{-i \langle p | W_r v_s \rangle t} (U(rs)f)(p). \end{aligned}$$

Comparing these two equations we can find that

$$((U_t(r)U_t(s))f)(p) = \phi(r, s, t) \omega(r, s) (U_t(rs)f)(p) \quad (18a)$$

where $\omega(r, s)$ is the usual projective phase factor, while $\phi(r, s, t)$ is defined by:

$$\phi(r, s, t) = e^{-i\gamma \langle v_r | W_r v_s \rangle t} = e^{i\xi_t(r, s)} \quad (18b)$$

with

$$\xi_t(r, s) = -\gamma \langle v_r | W_r v_s \rangle t = -\gamma \xi_{0,t}(r, s) \quad (18c)$$

(18c) is a bilinear continuous function in r, s coordinates which satisfies equations (3), like every projective phase exponent.

A. Physical interpretation of the phase exponents

The phase exponent (9) of galilean group

$$\begin{aligned} \xi(r, s) &= \gamma \xi_0(r, s) = \\ &= \frac{1}{2} \gamma (\langle u_r | W_r v_s \rangle - \langle v_r | W_r u_s \rangle + \eta_s \langle v_r | W_r v_s \rangle) \end{aligned}$$

has the physical dimensions of a action, and so it can be interpreted like the action of the particle in the frame rs . Indeed, set $r = (1, 0, 0, u)$, $s = (0, 0, v, 0)$, then $\xi(r, s) = \frac{\gamma}{2} \langle u | v \rangle$, that it has the dimension of an action, and $rs = (1, 0, v, u)$ is the Galilean transformation that relates an inertial frame system to another with relative velocity v and origin of axes translated by u .

About the phase exponent (18c), we propose the following interpretation.

Let be $W_r v_s$ velocity of Σ_s in Σ_r ; so $\langle v_r | W_r v_s \rangle$ is the velocity of Σ_s along the motion direction of Σ_r . So $\xi_t(r, s)$ is the contribution of the two coordinate systems Σ_r, Σ_s to the total action of the particle.

For example: let be $r = (1, 0, v_r, 0)$, $s = (1, 0, v_s, 0)$ two elements of galilean group. Then

$$(U_t(r)U_t(s)f)(p) = e^{-i\gamma \langle v_r | v_s \rangle t} (U_t(rs)f)(p).$$

And the contribution of the two coordinate systems to the total action of the particle is:

$$-\gamma \langle v_r | v_s \rangle t.$$

In conclusion: In this work, we brief remind the theory of phase exponents of ray representation of Lie groups and the application on galilean group in $(3+1)$, $(2+1)$ and $(1+1)$ dimensions. Finally, we find the time dependance of the ray representations of galilean group (17): the action of this representation on physical state $f(p)$ is given by:

$$f'(p, t) = e^{-i\langle p | v_r \rangle t} (U(r)f)(p) \quad (19)$$

For example, if we use (17) on (10) we obtain:

$$\begin{aligned} \phi'(p, t) &= \\ e^{-i\langle p | v_r \rangle t} e^{-i(\langle p | u_r \rangle - \frac{\eta_r}{2\gamma} \langle p | p \rangle + \frac{\eta_r}{2} \gamma \langle v_r | v_r \rangle - \gamma \langle u_r | v_r \rangle)} \phi(r^{-1}p) \end{aligned} \quad (20)$$

The phase exponents in (20) represents the total action of the particle in Σ_{rs} frame.

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- ⁸Ulf Uhlhorn, "Representation of symmetry transformations in quantum mechanics," Arkiv for Fysik **30**, 307 (1963).
- ⁹Valentine Bargmann, Journal of Mathematical Physics **5**, 862 (1964).
- ¹⁰Let \mathcal{U}_r be a continuous ray representations of a group G . For all r in a suitably chosen neighborhood \mathcal{N}_0 of the identity e of G , one may select a strongly continuous set of representatives $U_r \in \mathcal{U}_r$. A set of such representatives operator will be called an *admissible* set of representatives.⁵.
- ¹¹From the study of the full Galilean group in $(3+1)$ dimensions, it is possible to obtain the Bargmann's super-selection rules¹⁸:
1) To different masses there correspond inequivalent multipliers and hence inequivalent ray representations;
2) $SO(3)$ has two kind of inequivalent multipliers; one for integer spin (unitary representations), a second kind for semi-integer spin (ray representations)^{16,17}.
- ¹²Let be $\psi(x, t)$ are the solutions of Schrödinger equation, then:
$$\psi(x, t) \sim (2\pi)^{-\frac{3}{2}} \int \phi(p) e^{-i\frac{1}{2\gamma} \langle p | p \rangle t - \langle p | x \rangle} d^3 p$$
- ¹³A. Messiah, *Quantum Mechanics, vol.1* (North-Holland Publishing Company, Amsterdam, 1975).
- ¹⁴Where a_3 is a real number connected to the Casimir element
$$C_3 = 2HZ_1 - 2KZ_2 - P^2$$
where H, K, P are time translations, nonrelativistic boosts and space translations generators, and Z_1 and Z_2 are the two central elements.
- ¹⁵The infinitesimal generators of (12) are:
$$\begin{aligned} H &= \frac{1}{2\gamma} (p_1^2 + p_2^2) & N_1 &= \gamma \frac{\partial}{\partial p_1} + i \frac{\lambda}{2\gamma} p_2 \\ M &= is + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2} & N_2 &= \gamma \frac{\partial}{\partial p_2} - i \frac{\lambda}{2\gamma} p_1 \\ P_i &= p_i \quad (i = 1, 2) \end{aligned}$$
The infinitesimal generators of (13) are:
$$\begin{aligned} H &= \frac{1}{2\gamma} (p_1^2 + p_2^2), \quad P_i = p_i, \quad M = is + p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2}, \\ N_i &= \gamma \frac{\partial}{\partial p_i} \quad (i = 1, 2) \end{aligned}$$
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